Published by Institute of Physics Publishing for SISSA

Received: *May 1, 2008*
Accepted: *May 11, 2008*
Published: *May 15, 2008*

The Zamolodchikov-Faddeev algebra for open strings attached to giant gravitons

Changrim Ahn

Department of Physics, Ewha Womans University, Dae-Hyun 11-1, Seoul 120-750, South Korea
E-mail: ahn@ewha.ac.kr

Rafael I. Nepomechie

Physics Department, University of Miami, P.O. Box 248046, Coral Gables, FL 33124 U.S.A.
E-mail: nepomechie@physics.miami.edu

Abstract: We extend the Zamolodchikov-Faddeev algebra for the superstring sigma model on $AdS_5 \times S^5$, which was formulated by Arutyunov, Frolov and Zamaklar, to the case of open strings attached to maximal giant gravitons, which was recently considered by Hofman and Maldacena. We obtain boundary S-matrices which satisfy the standard boundary Yang-Baxter equation.

Keywords: Duality in Gauge Field Theories, AdS-CFT Correspondence, Integrable Field Theories, Exact S-Matrix.

Contents

1. Introduction

A factorizable S-matrix [1, 2] describing the scattering of world-sheet excitations of the $AdS_5 \times S^5$ superstring sigma model [3] has been proposed by Arutyunov, Frolov and Zamaklar (AFZ) [4]. This S-matrix is closely related to the one found earlier by Beisert [5] describing the scattering of excitations of the dynamic spin chain corresponding to planar $\mathcal{N} = 4$ super Yang-Mills. However, the AFZ "string" S-matrix obeys the standard Yang-Baxter equation, while Beisert's S-matrix obeys a twisted (dynamical) Yang-Baxter equation.[1] The string S-matrix (up to a phase) follows directly from the assumption that the excitations are described by a Zamolodchikov-Faddeev (ZF) algebra, and that they have a centrally extended $su(2|2) \oplus su(2|2)$ symmetry [5, 6]. It agrees with perturbative results obtained by direct computations [7].

Hofman and Maldacena (HM) [8] recently considered open strings attached to maximal giant gravitons [9] in $AdS_5 \times S^5$. (Related earlier work includes [10−12].) They proposed boundary S-matrices describing the reflection of world-sheet excitations (giant magnons) for two cases, namely, the $Y = 0$ and $Z = 0$ giant graviton branes. However, we have found that the boundary S-matrix for the latter case does not satisfy the standard boundary Yang-Baxter equation (BYBE) [13, 14].

[1] There are in fact three relevant S-matrices: $S^{\text{string}}_{\text{AFZ}}$, which is in the "string" basis, and satisfies the standard YBE; $S^{\text{chain}}_{\text{AFZ}}$, which is in the "spin chain" basis, and satisfies a twisted YBE; and S_{Beisert}, which is related to $S^{\text{chain}}_{\text{AFZ}}$ by the final (unnumbered) equation of section 7 in [4].

The purpose of this note is to construct related boundary S-matrices which do obey the standard BYBE. To this end, we extend the ZF algebra which was formulated by AFZ by introducing boundary operators with suitable symmetry properties. We explicitly verify that the resulting boundary S-matrices are indeed solutions of the standard BYBE.

The outline of this paper is as follows. In section 2 we briefly review the bulk ZF algebra and the computation of the bulk S-matrix, which in fact is the transpose of the matrix given in [4]. In section 3 we formulate the boundary ZF algebra, and present boundary S-matrices for both the $Y = 0$ and $Z = 0$ giant graviton branes. In section 4 we derive crossing relations for the boundary S-matrices and solve for the corresponding scalar factors. We conclude in section 5 with a brief discussion of our results.

2. Bulk ZF algebra and S-matrix

In this section, we briefly review the bulk ZF algebra and the computation of the bulk S-matrix.[2] Following AFZ [4], we denote the ZF operators by $A_i^\dagger(p)$, $i = 1\,,2\,,3\,,4$. These operators create asymptotic particle states of momentum p when acting on the vacuum state $|0\rangle$. The bulk S-matrix elements $S_{i\,j}^{i'j'}(p_1, p_2)$ are defined by the relation

$$A_i^\dagger(p_1)\, A_j^\dagger(p_2) = S_{i\,j}^{i'j'}(p_1, p_2)\, A_{j'}^\dagger(p_2)\, A_{i'}^\dagger(p_1)\,, \tag{2.1}$$

where summation over repeated indices is always understood. It is convenient to arrange these matrix elements into a 16×16 matrix S as follows,

$$S = S_{i\,j}^{i'j'}\, e_{i\,i'} \otimes e_{j\,j'}\,, \tag{2.2}$$

where e_{ij} is the usual elementary 4×4 matrix whose (i, j) matrix element is 1, and all others are zero. Although (2.2) is the standard convention, AFZ use a different convention (see eq. (8.4) in [4]), such that our matrix S is the *transpose* of theirs.

As is well known [1], starting from $A_i^\dagger(p_1)\, A_j^\dagger(p_2)\, A_k^\dagger(p_3)$, one can arrive at linear combinations of $A_{k''}^\dagger(p_3)\, A_{j''}^\dagger(p_2)\, A_{i''}^\dagger(p_1)$ by applying the relation (2.1) three times, in two different ways. The consistency condition is the Yang-Baxter equation,

$$S_{12}(p_1, p_2)\, S_{13}(p_1, p_3)\, S_{23}(p_2, p_3) \;= S_{23}(p_2, p_3)\, S_{13}(p_1, p_3)\, S_{12}(p_1, p_2)\,. \tag{2.3}$$

We use the standard convention $S_{12} = S \otimes \mathbb{I}$, $S_{23} = \mathbb{I} \otimes S$, and $S_{13} = \mathcal{P}_{12}\, S_{23}\, \mathcal{P}_{12}$, where $\mathcal{P}_{12} = \mathcal{P} \otimes \mathbb{I}$, $\mathcal{P} = e_{i\,j} \otimes e_{j\,i}$ is the permutation matrix, and $\mathbb{I}$ is the four-dimensional identity matrix. The ZF algebra (2.1) also implies the bulk unitarity equation

$$S_{12}(p_1, p_2)\, S_{21}(p_2, p_1) = \mathbb{I}\,, \tag{2.4}$$

where $S_{21} = \mathcal{P}_{12}\, S_{12}\, \mathcal{P}_{12}$.

[2] We consider the S-matrix corresponding to a single copy of the centrally extended $su(2|2)$ algebra; the full S-matrix is a tensor product of two such S-matrices.

JHEP05(2008)059

For later reference, we note (as also discussed in [4]) that the conjugate operators $\left(A_i^\dagger(p)\right)^\dagger = A^i(p)$ obey

$$A^i(p_1)\, A^j(p_2) = S^{i\,j}_{i'j'}(p_1,p_2)\, A^{j'}(p_2)\, A^{i'}(p_1)\,, \tag{2.5}$$

which together with (2.1) implies the so-called physical unitarity condition $S_{21}(p_2,p_1) = S_{12}^\dagger(p_1,p_2)$, and therefore

$$S_{12}(p_1,p_2)\, S_{12}^\dagger(p_1,p_2) = \mathbb{I}\,. \tag{2.6}$$

The centrally extended $su(2|2)$ algebra consists of the rotation generators $L_a^{\ b}$, $R_\alpha^{\ \beta}$, the supersymmetry generators $Q_\alpha^{\ a}$, $Q_a^{\dagger\alpha}$, and the central elements $C\,, C^\dagger\,, H$. Latin indices $a\,, b\,,\ldots$ take values $\{1\,, 2\}$, while Greek indices $\alpha\,, \beta\,,\ldots$ take values $\{3\,, 4\}$. These generators have the following nontrivial commutation relations [4, 5, 15]

$$\begin{aligned}
&\left[L_a^{\ b}\,, J_c\right] = \delta_c^b J_a - \frac{1}{2}\delta_a^b J_c\,, &&\left[R_\alpha^{\ \beta}\,, J_\gamma\right] = \delta_\gamma^\beta J_\alpha - \frac{1}{2}\delta_\alpha^\beta J_\gamma\,,\\
&\left[L_a^{\ b}\,, J^c\right] = -\delta_a^c J^b + \frac{1}{2}\delta_a^b J^c\,, &&\left[R_\alpha^{\ \beta}\,, J^\gamma\right] = -\delta_\alpha^\gamma J^\beta + \frac{1}{2}\delta_\alpha^\beta J^\gamma\,,\\
&\left\{Q_\alpha^{\ a}\,, Q_\beta^{\ b}\right\} = \epsilon_{\alpha\beta}\epsilon^{ab} C\,, &&\left\{Q_a^{\dagger\alpha}\,, Q_b^{\dagger\beta}\right\} = \epsilon^{\alpha\beta}\epsilon_{ab} C^\dagger\,,\\
&\left\{Q_\alpha^{\ a}\,, Q_b^{\dagger\beta}\right\} = \delta_b^a R_\alpha^{\ \beta} + \delta_\alpha^\beta L_b^{\ a} + \frac{1}{2}\delta_b^a\delta_\alpha^\beta H\,,
\end{aligned} \tag{2.7}$$

where J_i (J^i) denotes any lower (upper) index of a generator, respectively.

The action of the bosonic generators on the ZF operators is given by

$$\begin{aligned}
L_a^{\ b}\, A_c^\dagger(p) &= \left(\delta_c^b\delta_a^d - \frac{1}{2}\delta_a^b\delta_c^d\right) A_d^\dagger(p) + A_c^\dagger(p)\, L_a^{\ b}\,, \qquad L_a^{\ b}\, A_\gamma^\dagger(p) = A_\gamma^\dagger(p)\, L_a^{\ b}\,,\\
R_\alpha^{\ \beta}\, A_\gamma^\dagger(p) &= \left(\delta_\gamma^\beta\delta_\alpha^\delta - \frac{1}{2}\delta_\alpha^\beta\delta_\gamma^\delta\right) A_\delta^\dagger(p) + A_\gamma^\dagger(p)\, R_\alpha^{\ \beta}\,, \qquad R_\alpha^{\ \beta}\, A_c^\dagger(p) = A_c^\dagger(p)\, R_\alpha^{\ \beta}\,.
\end{aligned} \tag{2.8}$$

Moreover, the action of the supersymmetry generators is given by (see eq. (4.21) in [4])

$$\begin{aligned}
Q_\alpha^{\ a}\, A_b^\dagger(p) &= e^{-ip/2}\left[a(p)\delta_b^a A_\alpha^\dagger(p) + A_b^\dagger(p)\, Q_\alpha^{\ a}\right]\,,\\
Q_\alpha^{\ a}\, A_\beta^\dagger(p) &= e^{-ip/2}\left[b(p)\epsilon_{\alpha\beta}\epsilon^{ab} A_b^\dagger(p) - A_\beta^\dagger(p)\, Q_\alpha^{\ a}\right]\,,\\
Q_a^{\dagger\alpha}\, A_b^\dagger(p) &= e^{ip/2}\left[c(p)\epsilon_{ab}\epsilon^{\alpha\beta} A_\beta^\dagger(p) + A_b^\dagger(p)\, Q_a^{\dagger\alpha}\right]\,,\\
Q_a^{\dagger\alpha}\, A_\beta^\dagger(p) &= e^{ip/2}\left[d(p)\delta_\beta^\alpha A_a^\dagger(p) - A_\beta^\dagger(p)\, Q_a^{\dagger\alpha}\right]\,.
\end{aligned} \tag{2.9}$$

AFZ work with a different set of relations for the supersymmetry generators which involve the world-sheet momentum operator (see eq. (4.15) in [4]). However, as we shall see in section 3.2, the relations (2.9) are more natural when dealing with a boundary.

The one-particle states $A_i^\dagger(p)|0\rangle$ operators form a representation of the symmetry algebra with $C = a\,b\,e^{-ip}\,,\quad C^* = c\,d\,e^{ip}\,,\ H = ad+bc$, provided $ad-bc = 1$. The representation is also unitary provided $d = a^*\,, c = b^*$. Since $C = ig(1-e^{-ip})$ [4], the parameters can be chosen as follows [4, 5]

$$a = \sqrt{g}\eta\,, \quad b = \sqrt{g}\frac{i}{\eta}\left(\frac{x^+}{x^-}-1\right)\,, \quad c = -\sqrt{g}\frac{\eta}{x^+}\,, \quad d = \sqrt{g}\frac{x^+}{i\eta}\left(1-\frac{x^-}{x^+}\right)\,, \tag{2.10}$$

where

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}\,, \quad \frac{x^+}{x^-} = e^{ip}\,, \quad \eta = \sqrt{i(x^- - x^+)}\,. \tag{2.11}$$

Hence,

$$H = -ig\left(x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-}\right)\,. \tag{2.12}$$

The S-matrix can be determined (up to a phase) by demanding that it commute with the symmetry generators.[3] That is, starting from $J\, A_i^\dagger(p_1)\, A_j^\dagger(p_2)|0\rangle$ where J is a symmetry generator, and assuming that J annihilates the vacuum state, one can arrive at linear combinations of $A_{j'}^\dagger(p_2)\, A_{i'}^\dagger(p_1)|0\rangle$ in two different ways, by applying the ZF relation (2.1) and the symmetry relations (2.8), (2.9) in different orders. The consistency condition is a system of linear equations for the S-matrix elements. The result for the nonzero matrix elements is [4]

$$\begin{aligned}
&S_{a\,a}^{a\,a} = A\,, && S_{\alpha\,\alpha}^{\alpha\,\alpha} = D\,, \\
&S_{a\,b}^{a\,b} = \frac{1}{2}(A-B)\,, && S_{a\,b}^{b\,a} = \frac{1}{2}(A+B)\,, \\
&S_{\alpha\,\beta}^{\alpha\,\beta} = \frac{1}{2}(D-E)\,, && S_{\alpha\,\beta}^{\beta\,\alpha} = \frac{1}{2}(D+E)\,, \\
&S_{a\,b}^{\alpha\,\beta} = -\frac{1}{2}\epsilon_{ab}\epsilon^{\alpha\beta}\,C\,, && S_{\alpha\,\beta}^{a\,b} = -\frac{1}{2}\epsilon^{ab}\epsilon_{\alpha\beta}\,F\,, \\
&S_{a\,\alpha}^{a\,\alpha} = G\,, && S_{a\,\alpha}^{\alpha\,a} = H\,, \qquad S_{\alpha\,a}^{a\,\alpha} = K\,, \qquad S_{\alpha\,a}^{\alpha\,a} = L\,,
\end{aligned} \tag{2.13}$$

where $a\,,b \in \{1\,,2\}$ with $a \ne b$; $\alpha\,,\beta \in \{3\,,4\}$ with $\alpha \ne \beta$; and

$$\begin{aligned}
A &= S_0\frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\frac{\eta_1\eta_2}{\tilde{\eta}_1\tilde{\eta}_2}\,, \\
B &= -S_0\left[\frac{x_2^- - x_1^+}{x_2^+ - x_1^-} + 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\right]\frac{\eta_1\eta_2}{\tilde{\eta}_1\tilde{\eta}_2}\,, \\
C &= S_0\frac{2ix_1^- x_2^-(x_1^+ - x_2^+)\eta_1\eta_2}{x_1^+ x_2^+(x_1^- - x_2^+)(1 - x_1^- x_2^-)}\,, \qquad D = -S_0\,, \\
E &= S_0\left[1 - 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\right]\,, \\
F &= S_0\frac{2i(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^- x_2^-)\tilde{\eta}_1\tilde{\eta}_2}\,,
\end{aligned}$$

[3]The idea of using nonlocal (fractional-spin) integrals of motion to determine bulk S-matrices goes at least as far back as the works [16, 17]. This approach was extended to boundary S-matrices in [18].

$$G = S_0 \frac{(x_2^- - x_1^-)}{(x_2^+ - x_1^-)} \frac{\eta_1}{\tilde{\eta}_1}\,, \qquad H = S_0 \frac{(x_2^+ - x_2^-)}{(x_1^- - x_2^+)} \frac{\eta_1}{\tilde{\eta}_2}\,,$$
$$K = S_0 \frac{(x_1^+ - x_1^-)}{(x_1^- - x_2^+)} \frac{\eta_2}{\tilde{\eta}_1}\,, \qquad L = S_0 \frac{(x_1^+ - x_2^+)}{(x_1^- - x_2^+)} \frac{\eta_2}{\tilde{\eta}_2}\,, \tag{2.14}$$

where

$$x_i^\pm = x^\pm(p_i)\,, \quad \eta_1 = \eta(p_1) e^{ip_2/2}\,, \quad \eta_2 = \eta(p_2)\,, \quad \tilde{\eta}_1 = \eta(p_1)\,, \quad \tilde{\eta}_2 = \eta(p_2) e^{ip_1/2}\,, \tag{2.15}$$

and $\eta(p)$ is given in (2.11). This S-matrix satisfies the standard Yang-Baxter equation (2.3). It also satisfies the unitarity equation (2.4), provided that the scalar factor obey

$$S_0(p_1, p_2)\, S_0(p_2, p_1) = 1\,. \tag{2.16}$$

3. Boundary ZF algebra and S-matrix

We consider now the problem of scattering from a boundary. Following HM [8], we consider the cases of the $Y = 0$ and $Z = 0$ giant graviton branes, which we consider in turn.

3.1 $Y = 0$ giant graviton brane

In order to describe boundary scattering, we extend the bulk ZF algebra (2.1) by introducing appropriate boundary operators which create the boundary-theory vacuum state $|0\rangle_B$ from $|0\rangle$ [14]. Since there is no boundary degree of freedom for the $Y = 0$ giant graviton brane, the corresponding boundary operator is a scalar. For a right boundary, we introduce a right boundary operator B_R, and define the right boundary S-matrix by

$$A_i^\dagger(p)\, B_R = R_i^{R\,i'}(p)\, A_{i'}^\dagger(-p)\, B_R\,. \tag{3.1}$$

We arrange the S-matrix elements in the usual way into a matrix $R^R = R_i^{R\,i'}\, e_{i\,i'}$. Starting from $A_i^\dagger(p_1)\, A_j^\dagger(p_2)\, B_R$, one can arrive at linear combinations of $A_{i'''}^\dagger(-p_1)\, A_{j'''}^\dagger(-p_2)\, B_R$ by applying each of the relations (2.1) and (3.1) two times, in two different ways. The consistency condition is the right BYBE

$$S_{12}(p_1, p_2)\, R_1^R(p_1)\, S_{21}(p_2, -p_1)\, R_2^R(p_2) = R_2^R(p_2)\, S_{12}(p_1, -p_2)\, R_1^R(p_1)\, S_{21}(-p_2, -p_1) \tag{3.2}$$

The algebra (3.1) also implies the right boundary unitarity equation

$$R^R(p)\, R^R(-p) = \mathbb{I}\,. \tag{3.3}$$

We also assume, in analogy with the bulk case (2.6), the physical unitarity condition

$$R^R(p)\, R^R(p)^\dagger = \mathbb{I}\,. \tag{3.4}$$

For a left boundary, we introduce a left boundary operator B_L, and use the conjugate ZF operators $A^i(p)$ to define a left boundary S-matrix $R^L(p)$,[4]

$$B_L\, A^i(p) = R_{i'}^{L\,i}(p)\, B_L\, A^{i'}(-p)\,. \tag{3.6}$$

[4]One could try to instead use $A_i^\dagger(p)$ to define a left boundary S-matrix, namely $B_L\, A_i^\dagger(p) = R_i^{L\,i'}(p)\, B_L\, A_{i'}^\dagger(-p)$, which would instead obey (cf. (3.9))

$$R_1^L(p_1)\, S_{12}(-p_1, p_2)\, R_2^L(p_2)\, S_{21}(-p_2, -p_1) = S_{12}(p_1, p_2)\, R_2^L(p_2)\, S_{21}(-p_2, p_1)\, R_1^L(p_1)\,. \tag{3.5}$$

However, this left boundary S-matrix would *not* obey the natural relation (3.8).

If we identify B_L with $(B_R)^\dagger$, then (3.1) and (3.6) imply

$$R^L(p) = R^R(p)^\dagger \,. \tag{3.7}$$

Hence, it suffices to consider only the case of right boundary scattering. The unitarity conditions (3.3), (3.4) then imply the relation

$$R^L(p) = R^R(-p) \tag{3.8}$$

which was proposed by HM. We remark that, starting from $B_L\, A^i(p)\, A^j(p)$, and with the help of (2.5), one can derive the left BYBE

$$\begin{aligned} &R_1^{L\,t_1}(p_1)\, S_{12}^{t_1 t_2}(-p_1,p_2)\, R_2^{L\,t_2}(p_2)\, S_{21}^{t_1 t_2}(-p_2,-p_1) \\ &\qquad\qquad = S_{12}^{t_1 t_2}(p_1,p_2)\, R_2^{L\,t_2}(p_2)\, S_{21}^{t_1 t_2}(-p_2,p_1)\, R_1^{L\,t_1}(p_1)\,, \end{aligned} \tag{3.9}$$

where t_i denotes transposition in the $i^{\rm th}$ space. Taking the transpose in both spaces 1 and 2, interchanging spaces 1 and 2 (i.e., conjugating both sides with the permutation matrix $\mathcal{P}_{12}$), and relabeling $p_2 \mapsto -p_1\,, p_1 \mapsto -p_2$, we recover the right BYBE (3.2) with the identification (3.8).

Following HM, we proceed to determine the boundary S-matrix using the symmetry of the problem. The $Y=0$ giant graviton brane preserves only an $su(1|2)$ subalgebra [8], which includes (say) the supersymmetry generators $Q_\alpha^{\ 1}$ and $Q_1^{\dagger\alpha}$ with $\alpha \in \{3\,,4\}$. The right boundary S-matrix is diagonal, with matrix elements

$$R_1^{R\,1} = r_1\,, \quad R_2^{R\,2} = r_2\,, \quad R_3^{R\,3} = R_4^{R\,4} = r\,. \tag{3.10}$$

Using first (2.9) and then (3.1), we find

$$\begin{aligned} Q_3^{\ 1}\, A_1^\dagger(p)\, B_R|0\rangle &= e^{-ip/2}\left[a(p) A_3^\dagger(p) + A_1^\dagger(p)\, Q_3^{\ 1}\right] B_R|0\rangle \\ &= e^{-ip/2} a(p) r A_3^\dagger(-p) B_R|0\rangle\,, \end{aligned} \tag{3.11}$$

where we have passed to the second equality using also the assumption that $Q_3^{\ 1}$ annihilates the vacuum state $B_R|0\rangle$. Reversing the order, i.e., using first (3.1) and then (2.9), we obtain

$$\begin{aligned} Q_3^{\ 1}\, A_1^\dagger(p)\, B_R|0\rangle &= r_1 Q_3^{\ 1} A_1^\dagger(-p) B_R|0\rangle = r_1 e^{ip/2}\left[a(-p) A_3^\dagger(-p) + A_1^\dagger(-p) Q_3^{\ 1}\right] B_R|0\rangle \\ &= r_1 e^{ip/2} a(-p) A_3^\dagger(-p) B_R|0\rangle\,. \end{aligned} \tag{3.12}$$

Consistency of the results (3.11) and (3.12) requires

$$\frac{r_1}{r} = e^{-ip}\frac{a(p)}{a(-p)} = e^{-ip}\,, \tag{3.13}$$

where, in passing to the second equality, we have used [8]

$$x^\pm(-p) = -x^\mp(p)\,, \qquad \eta(-p) = \eta(p)\,, \tag{3.14}$$

since $x^{\pm} \mapsto -x^{\mp}$ corresponds to $p \mapsto -p$, $H \mapsto H$. Similarly, starting from $Q_3{}^1\, A_4^\dagger(p)\, B_R|0\rangle$, we readily obtain

$$\frac{r_2}{r} = e^{ip}\frac{b(-p)}{b(p)} = -1\,. \tag{3.15}$$

The same results are obtained using instead the other conserved supersymmetry generators. We conclude that the right boundary S-matrix is given by the diagonal matrix[5]

$$R^R(p) = R_0^R(p)\,\mathrm{diag}(e^{-ip}\,, -1\,, 1\,, 1)\,. \tag{3.16}$$

We have explicitly verified that this matrix satisfies the standard BYBE (3.2). It also evidently satisfies the boundary unitarity equation (3.3), provided that the corresponding scalar factor satisfies

$$R_0^R(p)\, R_0^R(-p) = 1\,. \tag{3.17}$$

If we demand the conservation of the supersymmetry generators $Q_\alpha{}^2\,, Q_2^{\dagger\alpha}$ instead of $Q_\alpha{}^1\,, Q_1^{\dagger\alpha}$, then we obtain the same result (3.16) except with the first two elements permuted.

The matrix (3.16) is similar (but not identical) to the right boundary S-matrix proposed by HM. The latter does not satisfy (3.2), but it does satisfy (3.5). We note that the left HM boundary S-matrix and our right boundary S-matrix are related by

$$R^L(p)_{\mathrm{HM}} = R^R(p)\ \mathrm{U}(2p) \tag{3.18}$$

(up to a permutation of the first two elements), where $\mathrm{U}(p)$ is a diagonal matrix relating the "string" and "chain" bases given by (see eq. (8.8) in [4])

$$\mathrm{U}(p) = \mathrm{diag}(e^{ip/2}\,, e^{ip/2}\,, 1\,, 1)\,. \tag{3.19}$$

One can show that the boundary S-matrix (3.16) is essentially (i.e., up to permutations, etc.) the unique diagonal solution of the BYBE (3.2) with the AFZ bulk S-matrix. In particular, no free boundary parameters appear in the solution. This is different from the case of the Hubbard model [19], for which the BYBE has diagonal solutions with a free parameter [20]. This difference seems paradoxical, given that the AFZ S-matrix is related [21] to Shastry's R-matrix. This difference can be attributed to the fact that a specific parametrization of $x^{\pm}(p)$ is needed to relate the bulk matrices (see eqs. (12), (14) and (A.3) in [21]), which is incompatible with the boundary matrices in [20].

3.2 $Z = 0$ giant graviton brane

According to HM, the $Z = 0$ giant graviton brane has a boundary degree of freedom and full $su(2|2)$ symmetry. Correspondingly, we introduce a right boundary operator with an index $B_{j\,R}$,

$$A_i^\dagger(p)\, B_{j\,R} = R^{R\,i'j'}_{\ \,i\,j}(p)\, A_{i'}^\dagger(-p)\, B_{j'\,R}\,, \tag{3.20}$$

[5]The left boundary S-matrix (3.6) can be computed in a completely analogous manner using the Hermitian conjugate of the relations (2.9) with $(Q_\alpha{}^a)^\dagger = Q_a^{\dagger\alpha}$. The result is an accord with (3.8).

and we arrange the boundary S-matrix elements into the 16×16 matrix R^R,

$$R^R = R^{R\, i' j'}_{\ \ i\, j}\, e_{i\, i'} \otimes e_{j\, j'}\,. \tag{3.21}$$

It satisfies the right BYBE (cf. eq. (3.2))

$$\begin{aligned} &S_{12}(p_1, p_2)\, R^R_{13}(p_1)\, S_{21}(p_2, -p_1)\, R^R_{23}(p_2) \\ &\qquad = R^R_{23}(p_2)\, S_{12}(p_1, -p_2)\, R^R_{13}(p_1)\, S_{21}(-p_2, -p_1)\,, \end{aligned} \tag{3.22}$$

and the right boundary unitarity equation (3.3), where now $\mathbb{I}$ is the 16-dimensional identity matrix.

Moreover, we introduce the left boundary operator $B^i_L = (B_{i\, R})^\dagger$, and define the left boundary S-matrix by

$$B^i_L\, A^j(p) = R^{L\, i\, j}_{\ \ i' j'}(p)\, B^{i'}_L\, A^{j'}(-p)\,. \tag{3.23}$$

It follows from (3.20) and (3.23) that

$$R^L_{12}(p) = R^R_{21}(p)^{t_1 t_2\, *} \equiv R^R_{21}(p)^\dagger\,. \tag{3.24}$$

The unitarity conditions (3.3), (3.4) then imply a relation analogous to the one for the $Y = 0$ case (3.8),

$$R^L_{12}(p) = R^R_{21}(-p)\,. \tag{3.25}$$

We again use symmetry to compute the boundary S-matrix. We assume that the symmetry generators act on the right boundary operators as follows

$$\begin{aligned} {L_a}^b\, B_{c\, R} &= \left(\delta^b_c \delta^d_a - \frac{1}{2}\delta^b_a \delta^d_c\right) B_{d\, R}\,, \qquad & {L_a}^b\, B_{\gamma\, R} &= 0\,, \\ {R_\alpha}^\beta\, B_{\gamma\, R} &= \left(\delta^\beta_\gamma \delta^\delta_\alpha - \frac{1}{2}\delta^\beta_\alpha \delta^\delta_\gamma\right) B_{\delta\, R}\,, \qquad & {R_\alpha}^\beta\, B_{c\, R} &= 0\,, \end{aligned} \tag{3.26}$$

and[6]

$$\begin{aligned} {Q_\alpha}^a\, B_{b\, R} &= a_B \delta^a_b B_{\alpha\, R}\,, \\ {Q_\alpha}^a\, B_{\beta\, R} &= b_B \epsilon_{\alpha\beta} \epsilon^{ab} B_{b\, R}\,, \\ Q^{\dagger\alpha}_a\, B_{b\, R} &= c_B \epsilon_{ab} \epsilon^{\alpha\beta} B_{\beta\, R}\,, \\ Q^{\dagger\alpha}_a\, B_{\beta\, R} &= d_B \delta^\alpha_\beta B_{a\, R}\,. \end{aligned} \tag{3.27}$$

The boundary operators form a fundamental representation of the symmetry algebra (2.7) provided

$$a_B d_B - b_B c_B = 1\,, \tag{3.28}$$

[6] If we had used the commutation relations of the ZF operators with the supersymmetry generators preferred by AFZ (namely, eq. (4.15) in [4]) instead of (2.9), then operators $e^{\pm iP/2}$ would appear on the r.h.s. of (3.27).

JHEP05(2008)059

with

$$C = a_B b_B \,, \quad C^* = c_B d_B \,, \quad H = a_B d_B + b_B c_B \,. \tag{3.29}$$

We take $d_B = a_B^*\,, c_B = b_B^*$ (unitarity); and we set $C = ig$, which is consistent with the requirement $|C| = g$ [8]. A suitable parametrization is

$$a_B = \sqrt{g}\eta_B \,, \quad b_B = \sqrt{g}\frac{i}{\eta_B} \,, \quad c_B = \sqrt{g}\frac{\eta_B}{x_B} \,, \quad d_B = \sqrt{g}\frac{x_B}{i\eta_B} \,, \tag{3.30}$$

where

$$\eta_B = \sqrt{-ix_B} \,, \qquad x_B = \frac{i}{2g}\left(1 + \sqrt{1+4g^2}\right) . \tag{3.31}$$

This parametrization coincides with the one used by HM for a particular value of their parameter f_B, namely $f_B = i$. (See eqs. (3.34) - (3.37) in [8].) We emphasize that our parameters (3.30) are independent of p, in keeping with the fact that momentum is a property only of the bulk excitations. In contrast, because HM use Beisert's "non local" notation (see the second reference in [5]), their values of f_B are functions of p which change under scattering.

The nonzero matrix elements of the right boundary S-matrix are

$$\begin{aligned}
&R^{R\,a\,a}_{\;\,a\,a} = A \,, && R^{R\,\alpha\,\alpha}_{\;\,\alpha\,\alpha} = D \,, \\
&R^{R\,a\,b}_{\;\,a\,b} = \frac{1}{2}(A+B) \,, && R^{R\,b\,a}_{\;\,a\,b} = \frac{1}{2}(A-B) \,, \\
&R^{R\,\alpha\,\beta}_{\;\,\alpha\,\beta} = \frac{1}{2}(D+E) \,, && R^{R\,\beta\,\alpha}_{\;\,\alpha\,\beta} = \frac{1}{2}(D-E) \,, \\
&R^{R\,\alpha\,\beta}_{\;\,a\,b} = \frac{1}{2}\epsilon_{ab}\epsilon^{\alpha\beta}\,C \,, && R^{R\,a\,b}_{\;\,\alpha\,\beta} = \frac{1}{2}\epsilon^{ab}\epsilon_{\alpha\beta}\,F \,, \\
&R^{R\,a\,\alpha}_{\;\,a\,\alpha} = K \,, && R^{R\,\alpha\,a}_{\;\,a\,\alpha} = L \,, \qquad R^{R\,a\,\alpha}_{\;\,\alpha\,a} = G \,, \qquad R^{R\,\alpha\,a}_{\;\,\alpha\,a} = H \,,
\end{aligned} \tag{3.32}$$

where $a\,,b \in \{1\,,2\}$ with $a \ne b$; and $\alpha\,,\beta \in \{3\,,4\}$ with $\alpha \ne \beta$. Proceeding as before, we obtain

$$\begin{aligned}
A &= e^{-2ip}A_{\mathrm{HM}} = R_0^R\frac{x^-(x^+ + x_B)}{x^+(x^- - x_B)} \,, \\
B &= e^{-2ip}B_{\mathrm{HM}} = R_0^R\frac{2x^+x^-x_B + (x^+ - x_B)[-2(x^+)^2 + 2(x^-)^2 + x^+x^-]}{(x^+)^2(x^- - x_B)} \,, \\
C &= C_{\mathrm{HM}} = R_0^R\frac{2\eta\eta_B}{i}\frac{(x^- + x^+)(x^-x_B - x^+x_B - x^-x^+)}{x_Bx^-(x^+)^2(x^- - x_B)} \,, \qquad D = D_{\mathrm{HM}} = R_0^R \,, \\
E &= E_{\mathrm{HM}} = R_0^R\frac{2[(x^+)^2-(x^-)^2][-x^+x^-+x_B(x^--x^++x^-(x^+)^2]-x_B(x^+x^-)^2(x_B-x^-)}{(x^-x^+)^2x_B(x^--x_B)} \,, \\
F &= e^{-2ip}F_{\mathrm{HM}} = R_0^R\frac{2i}{\eta\eta_B}\frac{[(x^+)^2 - (x^-)^2](x_Bx^+ - x_Bx^- + x^+x^-)}{(x^+)^2x^-(x^- - x_B)} \,, \\
G &= e^{-ip}G_{\mathrm{HM}} = R_0^R\frac{\eta_B}{\eta}\frac{(x^+)^2 - (x^-)^2}{x^+(x^- - x_B)} \,, \qquad H = e^{-ip}H_{\mathrm{HM}} = R_0^R\frac{(x^+)^2 - x_Bx^-}{x^+(x^- - x_B)} \,,
\end{aligned}$$

$$K = e^{-ip} K_{\mathrm{HM}} = R_0^R \frac{(x^-)^2 + x_B x^+}{x^+(x^- - x_B)}\,, \qquad L = e^{-ip} L_{\mathrm{HM}} = R_0^R \frac{\eta}{\eta_B} \frac{(x^+ + x^-) x_B}{x^+(x^- - x_B)}\,, \tag{3.33}$$

where A_{HM}, etc. are the corresponding HM amplitudes for the *left* boundary S-matrix (see eq. (3.46) in [8]) with $f = i$. We have explicitly verified that the right BYBE (3.22) is satisfied, as well as the boundary unitarity equation (3.3), provided that the scalar factor obey (3.17).

We note that the left HM boundary S-matrix and our right boundary S-matrix are related by (cf. eq. (3.18))

$$R^L(p)_{\mathrm{HM}} = R^R(p)\ \mathrm{U}(2p) \otimes \mathrm{U}(2p)\,, \tag{3.34}$$

where $\mathrm{U}(p)$ is given by (3.19).

4. Crossing relations and scalar factors

We turn now to the derivation of crossing relations, which (together with the unitarity relations) help determine the scalar factors of the S-matrices. For the boundary S-matrices, the crossing relations and scalar factors are similar to (but not the same as) those for the HM boundary S-matrices.

4.1 Bulk

For the bulk S-matrix, a crossing relation was first proposed by Janik [22] based on a Hopf algebra structure of the symmetry algebra. AFZ subsequently gave an alternative derivation of the crossing relation based on the ZF algebra. We now reformulate in terms of ZF operators yet another derivation of the crossing relation, due to Beisert [5], which is particularly convenient to generalize to the boundary case [8]. To this end, we define the "singlet" operator

$$I(p) = C^{ij}(p)\, A_i^\dagger(p)\, A_j^\dagger(\bar{p}) \equiv \mathfrak{c}(p)\, \epsilon^{ab} A_a^\dagger(p)\, A_b^\dagger(\bar{p}) + \epsilon^{\alpha\beta} A_\alpha^\dagger(p)\, A_\beta^\dagger(\bar{p})\,, \tag{4.1}$$

where (as before) $a\,, b \in \{1\,, 2\}$, $\alpha\,, \beta \in \{3\,, 4\}$, and the function $\mathfrak{c}(p)$ is yet to be determined. Hence, $C(p)$ is the 4×4 matrix

$$C(p) = \begin{pmatrix} 0 & \mathfrak{c}(p) & 0 & 0 \\ -\mathfrak{c}(p) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{4.2}$$

Moreover, $\bar{p}$ denotes the antiparticle momentum, with [22, 4]

$$x^\pm(\bar{p}) = \frac{1}{x^\pm(p)}\,, \tag{4.3}$$

since $x^\pm \mapsto 1/x^\pm$ corresponds to $p \mapsto -p \equiv \bar{p}\,,\ H \mapsto -H \equiv \bar{H}$. One can readily check (with the help of eq. (2.8)) that the singlet operator commutes with the bosonic generators. The function $\mathfrak{c}(p)$ is determined by the condition that the singlet operator also commute with the supersymmetry generators. Indeed, the condition ${Q_3}^1 I(p)|0\rangle = I(p)\, {Q_3}^1|0\rangle = 0$ readily leads (with the help of eq. (2.9)) to

$$\mathfrak{c}(p) = e^{ip/2}\frac{b(\bar{p})}{a(p)} = -e^{-ip/2}\frac{b(p)}{a(\bar{p})} = -i\,\mathrm{sign}(p)\,. \tag{4.4}$$

This computation evidently parallels the one in AFZ for the charge conjugation matrix. However, the matrix (6.8) in [4] is proportional to our $C(-p)$.[7]

The crossing relation follows from the requirement that the singlet operator scatter trivially with a particle. Indeed,

$$\begin{aligned} A_i^\dagger(p_1)\, I(p_2) &= C^{jk}(p_2)\, A_i^\dagger(p_1)\, A_j^\dagger(p_2)\, A_k^\dagger(\bar{p}_2) \\ &= C^{jk}(p_2)\, S_{ij}^{i'j'}(p_1,p_2)\, A_{j'}^\dagger(p_2)\, A_{i'}^\dagger(p_1)\, A_k^\dagger(\bar{p}_2) \\ &= C^{jk}(p_2)\, S_{ij}^{i'j'}(p_1,p_2)\, S_{i'k}^{i''k'}(p_1,\bar{p}_2)\, A_{j'}^\dagger(p_2)\, A_{k'}^\dagger(\bar{p}_2)\, A_{i''}^\dagger(p_1) \\ &\equiv I(p_2)\, A_i^\dagger(p_1) \end{aligned} \tag{4.5}$$

implies the relation

$$C^{jk}(p_2)\, S_{ij}^{i'j'}(p_1,p_2)\, S_{i'k}^{i''k'}(p_1,\bar{p}_2) = C^{j'k'}(p_2)\, \delta_i^{i''}\,, \tag{4.6}$$

which can be re-expressed in matrix notation as

$$S_{12}^{t_2}(p_1,p_2)\, C_2(p_2)\, S_{12}(p_1,\bar{p}_2)\, C_2(p_2)^{-1} = \mathbb{I}\,. \tag{4.7}$$

Substituting the result (2.13), (2.14) for the S-matrix, we obtain a crossing relation for the bulk scalar factor

$$S_0(p_1,p_2)\, S_0(p_1,\bar{p}_2) = \frac{1}{f(p_1,p_2)}\,, \tag{4.8}$$

where [22]

$$f(p_1,p_2) = \frac{\left(\frac{1}{x_1^+} - x_2^-\right)(x_1^+ - x_2^+)}{\left(\frac{1}{x_1^-} - x_2^-\right)(x_1^- - x_2^+)}\,. \tag{4.9}$$

Similarly, by demanding $I(\bar{p}_1)\, A_k^\dagger(p_2) = A_k^\dagger(p_2)\, I(\bar{p}_1)$ and using the fact that the matrix $C(p)$ is antisymmetric, one can also formally obtain

$$S_{12}^{t_1}(p_1,p_2)\, C_1(\bar{p}_1)\, S_{12}(\bar{p}_1,p_2)\, C_1(\bar{p}_1)^{-1} = \mathbb{I}\,, \tag{4.10}$$

which implies a second crossing relation for the bulk scalar factor [4]

$$S_0(p_1,p_2)\, S_0(\bar{p}_1,p_2) = \frac{1}{f(p_1,p_2)}\,. \tag{4.11}$$

[7] In fact, the momentum dependence of the charge conjugation matrix is spurious and can be removed by properly resolving the branch cut ambiguity as noticed in [23].

The crossing equations (4.8), (4.11) corresponding to the AFZ (string) S-matrix are the same as Janik's relations [22] corresponding to Beisert's (spin chain) S-matrix [5], except the right-hand-sides are inverted. Correspondingly, the solutions are also inversely related.

In more detail, let us now now consider the full theory, for which there are two $su(2|2)$ factors. Setting [24, 25][8]

$$S_0(p_1\,,p_2)^2 = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\frac{1-\frac{1}{x_1^+ x_2^-}}{1-\frac{1}{x_1^- x_2^+}}\sigma(p_1\,,p_2)^2\,, \tag{4.12}$$

the crossing equations (4.8), (4.11) imply that the "dressing factor" $\sigma(p_1\,,p_2)$ obeys

$$\sigma(\bar{p}_1,p_2)\,\sigma(p_1,p_2) = \frac{x_2^-}{x_2^+}\frac{1}{f(p_1,p_2)}\,, \qquad \sigma(p_1,\bar{p}_2)\,\sigma(p_1,p_2) = \frac{x_1^+}{x_1^-}\frac{1}{f(p_1,p_2)}\,, \tag{4.13}$$

and the unitarity equation (2.16) implies

$$\sigma(p_1,p_2)\,\sigma(p_2,p_1) = 1\,. \tag{4.14}$$

The relations (4.13), (4.14) are "universal" in the sense that the dressing factor for the spin chain S-matrix obeys the same relations [26]. A solution is given by [25]–[28]

$$\sigma(x_1^\pm\,,x_2^\pm) = \frac{R(x_1^+\,,x_2^+)\;R(x_1^-\,,x_2^-)}{R(x_1^+\,,x_2^-)\;R(x_1^-\,,x_2^+)}\,, \qquad R(x_1\,,x_2) = e^{i[\chi(x_1\,,x_2)-\chi(x_2\,,x_1)]}\,, \tag{4.15}$$

where [28]

$$\chi(x_1\,,x_2) = -i\oint_{|z_1|=1}\frac{dz_1}{2\pi}\oint_{|z_2|=1}\frac{dz_2}{2\pi}\frac{\ln\Gamma\left(1+ig\left(z_1+\frac{1}{z_1}-z_2-\frac{1}{z_2}\right)\right)}{(x_1-z_1)(x_2-z_2)}\,. \tag{4.16}$$

4.2 Boundary: $Y=0$ giant graviton brane

For the boundary case, we follow HM and consider the scattering of the singlet operator (4.1) off the boundary. For the right boundary, we obtain

$$\begin{aligned}
I(p)\,B_R &= C^{ij}(p)\,A_i^\dagger(p)\,A_j^\dagger(\bar{p})\,B_R \\
&= C^{ij}(p)\,R_j^{R\,j'}(\bar{p})\,A_i^\dagger(p)\,A_{j'}^\dagger(-\bar{p})\,B_R \\
&= C^{ij}(p)\,R_j^{R\,j'}(\bar{p})\,S_{ij'}^{i'j''}(p,-\bar{p})\,A_{j''}^\dagger(-\bar{p})\,A_{i'}^\dagger(p)\,B_R \\
&= C^{ij}(p)\,R_j^{R\,j'}(\bar{p})\,S_{ij'}^{i'j''}(p,-\bar{p})\,R_{i'}^{R\,i''}(p)\,A_{j''}^\dagger(-\bar{p})\,A_{i''}^\dagger(-p)\,B_R \\
&\equiv I(-\bar{p})\,B_R\,,
\end{aligned} \tag{4.17}$$

which implies the relation

$$C^{ij}(p)\,R_j^{R\,j'}(\bar{p})\,S_{ij'}^{i'j''}(p,-\bar{p})\,R_{i'}^{R\,i''}(p) = C^{j''i''}(p)\,. \tag{4.18}$$

[8] For the spin chain S-matrix, the r.h.s. of (4.12) is inverted [8, 26].

Substituting the results for the bulk (2.13), (2.14) and boundary (3.16) S-matrices, we obtain the right boundary crossing relation

$$R_0^R(p)\, R_0^R(\bar{p})\, S_0(p,-\bar{p}) = \frac{1}{h_b(-p)} = h_b(p)\,, \tag{4.19}$$

where [8]

$$h_b(p) = \frac{\frac{1}{x^-} + x^-}{\frac{1}{x^+} + x^+}\,. \tag{4.20}$$

The boundary crossing relation (4.19) is similar to the one found by Ghoshal and Zamolodchikov [14] for relativistic integrable theories, and is the same as HM (3.29), except with $p \mapsto -p$ in the r.h.s. .

For the full theory, the crossing relation becomes

$$R_0^R(p)^2\, R_0^R(\bar{p})^2 = h_b(p)^2 \frac{1}{S_0(p,-\bar{p})^2} = h_b(p) \frac{1}{\sigma(p,-\bar{p})^2}\,, \tag{4.21}$$

where we have used (4.12). Since the r.h.s. is the inverse of HM's relation (3.31), the solution is the inverse of the solution found by Chen and Correa (see eq. (27) in [29])

$$R_0^R(p)^2 = R_0^R(p)_{\mathrm{HM}}^{-2} = F(p)\, \sigma(p\,,-p)\,, \tag{4.22}$$

where we have used (4.14), and $F(p)$ is a CDD-type factor obeying

$$F(p)\, F(\bar{p}) = 1\,, \qquad F(p)\, F(-p) = 1\,. \tag{4.23}$$

4.3 Boundary: $Z = 0$ giant graviton brane

For the right $Z = 0$ boundary, a calculation analogous to (4.17) implies the relation

$$C^{ij}(p)\, R^{R\, j'k'}_{jk}(\bar{p})\, S^{i'j''}_{ij'}(p,-\bar{p})\, R^{R\, i''k''}_{i'k'}(p) = C^{j''i''}(p)\, \delta^{k''}_{k}\,. \tag{4.24}$$

Substituting the results for the bulk (2.13), (2.14) and boundary (3.16) S-matrices, we obtain the right boundary crossing relation

$$R_0^R(p)\, R_0^R(\bar{p})\, S_0(p,-\bar{p}) = \frac{1}{h_b(-p) h_B(-p)} = \frac{h_b(p)}{h_B(-p)}\,, \tag{4.25}$$

where [29, 30]

$$\begin{aligned} h_B(p) &= \frac{x^+}{x^-}\left(\frac{x_B - x^-}{x_B - x^+}\right) \frac{1 + (x_B x^- x^+)^2}{(1-(x_B x^+)^2)(1 - x^- x^+)} \\ &= \left(\frac{x_B - x^-}{x_B - x^+}\right)\left(\frac{\frac{1}{x^-} + x_B}{\frac{1}{x^+} + x_B}\right)\,. \end{aligned} \tag{4.26}$$

The boundary crossing relation (4.25) is the same as the one found in [29], except with $p \mapsto -p$ in the r.h.s. .

For the full theory, the crossing relation becomes

$$R_0^R(p)^2\, R_0^R(\bar{p})^2 = \frac{h_b(p)^2}{h_B(-p)^2}\frac{1}{S_0(p,-\bar{p})^2} = \frac{h_b(p)}{h_B(-p)^2}\frac{1}{\sigma(p,-\bar{p})^2}\,. \tag{4.27}$$

Comparing with the corresponding $Y = 0$ results (4.21), (4.22), we see that

$$R_0^R(p)^2 = F(p)\,\sigma(p\,,-p)\,\tilde{R}_0^R(p)^2\,, \tag{4.28}$$

where

$$\tilde{R}_0^R(p)^2\,\tilde{R}_0^R(\bar{p})^2 = \frac{1}{h_B(-p)^2}\,, \qquad \tilde{R}_0^R(p)^2\,\tilde{R}_0^R(-p)^2 = 1\,. \tag{4.29}$$

We solve for $\tilde{R}_0^R(p)^2$ following [30] using the identities

$$\sigma(p\,,-x_B)^2\,\sigma(\bar{p}\,,-x_B)^2 = \frac{h_b(p)^2}{h_B(-p)^2}\,, \qquad \sigma(p\,,-x_B)^2\,\sigma(-p\,,-x_B)^2 = 1\,, \tag{4.30}$$

which we prove in appendix A. We conclude that

$$\tilde{R}_0^R(p)^2 = \frac{1}{h_b(p)}\sigma(p\,,-x_B)^2\,. \tag{4.31}$$

As noted by HM, the boundary S-matrix for the full theory has a double pole at $x^- = x_B$ (see eq. (3.33) above). It can be reduced to a simple pole (corresponding to the second boundary bound state [8]) by choosing the CDD factor

$$F(p) = \left(\frac{x^- - x_B}{\frac{1}{x^-} - x_B}\right)\left(\frac{\frac{1}{x^+} + x_B}{x^+ + x_B}\right)\,, \tag{4.32}$$

which contains the factor $(x^- - x_B)$ and satisfies (4.23). Summarizing, the right boundary scalar factor $R_0^R(p)^2$ is given by (4.28), (4.31) and (4.32).

5. Discussion

We have seen that not only bulk [4] but also boundary S-matrices of string/gauge theory can satisfy the usual Yang-Baxter equation. The latter are closely related to the boundary S-matrices which were proposed in [8], as can be seen from eqs. (3.18) and (3.34). Presumably, as in the bulk case, the differences are due to working in different bases. It should now be possible to bring the well-developed techniques of the Quantum Inverse Scattering Method to bear on boundary problems in string/gauge theory. For example, one can now try to construct the commuting "double-row" transfer matrix [31] and determine its eigenvalues in terms of roots of corresponding Bethe Ansatz equations. We hope to be able to address these and related problems in the near future.

Acknowledgments

This work was initiated at the 2007 APCTP Focus Program "Liouville, Integrability and Branes (4)". We thank the participants, and also A. Belitsky, for discussions. We are also grateful to G. Arutyunov and D. Hofman for reading and commenting on a draft. This work was supported in part by KRF-2007-313-C00150 (CA) and by the National Science Foundation under Grants PHY-0244261 and PHY-0554821 (RN).

A. Derivation of (4.30)

In order to derive the first identity in (4.30), we first derive the more general result[9]

$$\sigma(y\,, x_{(n)})^2\, \sigma(\bar{y}\,, x_{(n)})^2 = \left(\frac{x^-_{(n)}}{x^+_{(n)}}\right)^2 \frac{h(y\,, x_{(n)})^2}{f(y\,, x_{(n)})^2}\,, \tag{A.1}$$

where (cf. (4.9))

$$f(y\,, x_{(n)}) = \frac{\left(\frac{1}{y^+} - x^-_{(n)}\right)\left(y^+ - x^+_{(n)}\right)}{\left(\frac{1}{y^-} - x^-_{(n)}\right)\left(y^- - x^+_{(n)}\right)}\,, \qquad h(y\,, x_{(n)}) = \frac{y^+ + \frac{1}{y^+} - x^+_{(n)} - \frac{1}{x^+_{(n)}}}{y^- + \frac{1}{y^-} - x^-_{(n)} - \frac{1}{x^-_{(n)}}}\,. \tag{A.2}$$

Moreover, $x^\pm_{(n)}$ are the parameters corresponding to an n-magnon bound state of momentum p given by [28, 32]

$$x^\pm_{(n)} = \frac{e^{\pm ip/2}}{4g\sin(p/2)}\left(n + \sqrt{n^2 + 16g^2\sin^2(p/2)}\right)\,, \tag{A.3}$$

which obey the constraint

$$x^+_{(n)} + \frac{1}{x^+_{(n)}} - x^-_{(n)} - \frac{1}{x^-_{(n)}} = \frac{in}{g}\,. \tag{A.4}$$

The n magnons have momenta $p_1\,, p_2\,, \ldots\,, p_n$ which form a composite (Bethe n-string), with

$$x^-_j = x^+_{j-1}\,, \qquad j = 2\,, \ldots\,, n\,, \tag{A.5}$$

where $x^\pm_j \equiv x^\pm(p_j)$. Indeed, since

$$x^+_j + \frac{1}{x^+_j} - x^-_j - \frac{1}{x^-_j} = \frac{i}{g}\,, \qquad j = 1\,, \ldots\,, n\,, \tag{A.6}$$

summing over j yields the constraint (A.4), where

$$x^+_{(n)} = x^+_n\,, \qquad x^-_{(n)} = x^-_1\,. \tag{A.7}$$

With the help of (4.15), (A.5), we obtain

$$\prod_{j=1}^n \sigma(y\,, x_j) = \prod_{j=1}^n \frac{R(y^+\,, x^+_j)R(y^-\,, x^-_j)}{R(y^+\,, x^-_j)R(y^-\,, x^+_j)} = \frac{R\left(y^+\,, x^+_{(n)}\right) R\left(y^-\,, x^-_{(n)}\right)}{R\left(y^+\,, x^-_{(n)}\right) R\left(y^-\,, x^+_{(n)}\right)} \equiv \sigma\left(y\,, x_{(n)}\right). \tag{A.8}$$

The l.h.s. of (A.1) is therefore given by

$$\begin{aligned}
\sigma(y\,, x_{(n)})^2\, \sigma(\bar{y}\,, x_{(n)})^2 &= \prod_{j=1}^n \left[\sigma(y\,, x_j)\sigma(\bar{y}\,, x_j)\right]^2 = \prod_{j=1}^n \left[\frac{x^-_j}{x^+_j}\frac{1}{f(y, x_j)}\right]^2 \\
&= \left(\frac{x^-_{(n)}}{x^+_{(n)}}\right)^2 \prod_{j=1}^n \frac{1}{f(y, x_j)^2}\,,
\end{aligned} \tag{A.9}$$

[9] We denote the momentum dependence of functions by $x\,, x^\pm\,, p$ (or $y\,, y^\pm\,, p$, etc.) interchangeably.

where we have used (4.13), as well as the relation

$$\prod_{j=1}^{n} \frac{x_j^-}{x_j^+} = \frac{x_{(n)}^-}{x_{(n)}^+}\,, \tag{A.10}$$

which follows from (A.5). In order to evaluate the remaining product in (A.9), we make use of the decomposition [26]

$$f(y\,,x)^2 = \left[\frac{f(y\,,x)}{f(\bar{y}\,,x)}\right]\left[f(y\,,x)f(\bar{y}\,,x)\right] \equiv \alpha(y\,,x)\ \beta(y\,,x)\,. \tag{A.11}$$

Recalling the definition (4.9), we obtain

$$\begin{aligned}
\alpha(y\,,x) &= \frac{f(y\,,x)}{f(\bar{y}\,,x)} = \left(\frac{y^+ - x^+}{y^+ - x^-}\right)\left(\frac{y^- - x^-}{y^- - x^+}\right)\left(\frac{y^- - \frac{1}{x^+}}{y^- - \frac{1}{x^-}}\right)\left(\frac{y^+ - \frac{1}{x^-}}{y^+ - \frac{1}{x^+}}\right)\,, \\
\beta(y\,,x) &= f(y\,,x)f(\bar{y}\,,x) = \frac{u(y) - u(x) + \frac{i}{g}}{u(y) - u(x) - \frac{i}{g}}\,,
\end{aligned} \tag{A.12}$$

where $u(x)$ is defined as [26]

$$u(x) = x^+ + \frac{1}{x^+} - \frac{i}{2g} = x^- + \frac{1}{x^-} + \frac{i}{2g}\,. \tag{A.13}$$

Note that

$$u(x_j) = u(x_{j-1}) + \frac{i}{g}\,. \tag{A.14}$$

After some algebra, we obtain

$$\prod_{j=1}^{n} \alpha(y\,,x_j) = \left(\frac{y^+ - x_{(n)}^+)}{y^+ - x_{(n)}^-)}\right)\left(\frac{y^- - x_{(n)}^-)}{y^- - x_{(n)}^+)}\right)\left(\frac{y^- - \frac{1}{x_{(n)}^+)}}{y^- - \frac{1}{x_{(n)}^-)}}\right)\left(\frac{y^+ - \frac{1}{x_{(n)}^-)}}{y^+ - \frac{1}{x_{(n)}^+)}}\right)\;; \tag{A.15}$$

and, using (A.14),

$$\begin{aligned}
\prod_{j=1}^{n} \beta(y\,,x_j) &= \left(\frac{u(y) - u(x_1) + \frac{i}{g}}{u(y) - u(x_n) - \frac{i}{g}}\right)\left(\frac{u(y) - u(x_1)}{u(y) - u(x_n)}\right) \\
&= \left(\frac{y^+ - x_{(n)}^-}{y^- - x_{(n)}^+}\right)\left(\frac{1 - \frac{1}{y^+ x_{(n)}^-}}{1 - \frac{1}{y^- x_{(n)}^+}}\right)\frac{1}{h(y\,,x_{(n)})}\,,
\end{aligned} \tag{A.16}$$

where $h(y\,,x_{(n)})$ is defined in (A.2). Combining the results (A.11), (A.15), (A.16), we eventually obtain

$$\prod_{j=1}^{n} f(y,x_j)^2 = \prod_{j=1}^{n} \alpha(y\,,x_j)\beta(y\,,x_j) = \frac{f(y\,,x_{(n)})^2}{h(y\,,x_{(n)})^2}\,, \tag{A.17}$$

where $f(y\,,x_{(n)})$ is defined in (A.2). Substituting this result into (A.9), we arrive at the desired result (A.1).

We are finally in a position to prove the first identity in (4.30). The key point [30] is that the boundary bound state can be regarded as an $n = 2$ magnon bound state with momentum $p = \pi$,

$$\pm x_B = x^{\pm}_{(2)}(p = \pi)\,, \tag{A.18}$$

as follows from (A.3) and the expression (3.31) for x_B. It follows from (A.1) that

$$\sigma(y\,, x_B)^2\,\sigma(\bar{y}\,, x_B)^2 = \frac{h_b(y)^2}{f(y\,, x_B)^2}\,, \tag{A.19}$$

where $\sigma(y\,, x_B) \equiv \sigma(y\,, x_{(2)}(p = \pi))$ (see eq. (A.8)). Moreover, recalling (A.2),

$$f(y\,, x_B) \equiv f(y\,, x_{(2)}(p = \pi)) = \frac{\left(\frac{1}{y^+} + x_B\right)(y^+ - x_B)}{\left(\frac{1}{y^-} + x_B\right)(y^- - x_B)}\,, \tag{A.20}$$

and, since $x_B + 1/x_B = i/g$,

$$h(y\,, x_{(2)}(p = \pi)) = \frac{y^- + \frac{1}{y^-}}{y^+ + \frac{1}{y^+}} = h_b(y)\,, \tag{A.21}$$

where h_b is defined in (4.20). Finally, performing in (A.19) the continuation $x_B \mapsto -x_B$, we obtain

$$\sigma(y\,, -x_B)^2\,\sigma(\bar{y}\,, -x_B)^2 = \frac{h_b(y)^2}{f(y\,, -x_B)^2} = \frac{h_b(p)^2}{h_B(-p)^2}\,. \tag{A.22}$$

The second equality follows from $f(y\,, -x_B) = h_B(-p)$, where $h_B(p)$ is given by (4.26). The result (A.22) is the first identity in (4.30).

The identity

$$\chi(x_1\,, x_2) = \chi(-x_2\,, -x_1) \tag{A.23}$$

follows from (4.16) by replacing $z_{1,2} \mapsto -z_{1,2}$ and interchanging $z_1 \leftrightarrow z_2$. It then follows from (4.15) that

$$R(x_1\,, x_2) = R(-x_2\,, -x_1)\,. \tag{A.24}$$

The second (unitarity) relation in (4.30) follows readily from (A.8), (A.18) and the identities (4.14), (A.24).

References

[1] A.B. Zamolodchikov and A.B. Zamolodchikov, *Factorized S matrices in two-dimensions as the exact solutions of certain relativistic quantum field models*, *Ann. Phys. (NY)* **120** (1979) 253.

[2] L.D. Faddeev, *Quantum completely integral models of field theory*, *Sov. Sci. Rev.* **C1** (1980) 107.

[3] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys.* **B 533** (1998) 109 [hep-th/9805028].

[4] G. Arutyunov, S. Frolov and M. Zamaklar, *The Zamolodchikov-Faddeev algebra for $AdS_5 \times S^5$ superstring*, *JHEP* **04** (2007) 002 [hep-th/0612229].

[5] N. Beisert, *The* SU(2|2) *dynamic S-matrix*, hep-th/0511082;
The analytic Bethe ansatz for a chain with centrally extended SU(2|2) *symmetry*, *J. Stat. Mech.* (2007) P01017 [nlin/0610017].

[6] G. Arutyunov, S. Frolov, J. Plefka and M. Zamaklar, *The off-shell symmetry algebra of the light-cone $AdS_5 \times S^5$ superstring*, *J. Phys.* **A 40** (2007) 3583 [hep-th/0609157].

[7] T. Klose, T. McLoughlin, R. Roiban and K. Zarembo, *Worldsheet scattering in $AdS_5 \times S^5$*, *JHEP* **03** (2007) 094 [hep-th/0611169];
T. Klose, T. McLoughlin, J.A. Minahan and K. Zarembo, *World-sheet scattering in $AdS_5 \times S^5$ at two loops*, *JHEP* **08** (2007) 051 [arXiv:0704.3891].

[8] D.M. Hofman and J.M. Maldacena, *Reflecting magnons*, *JHEP* **11** (2007) 063 [arXiv:0708.2272].

[9] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from anti-de Sitter space*, *JHEP* **06** (2000) 008 [hep-th/0003075];
M.T. Grisaru, R.C. Myers and O. Tafjord, *SUSY and Goliath*, *JHEP* **08** (2000) 040 [hep-th/0008015];
A. Hashimoto, S. Hirano and N. Itzhaki, *Large branes in AdS and their field theory dual*, *JHEP* **08** (2000) 051 [hep-th/0008016].

[10] D. Berenstein and S.E. Vazquez, *Integrable open spin chains from giant gravitons*, *JHEP* **06** (2005) 059 [hep-th/0501078].

[11] A. Agarwal, *Open spin chains in super Yang-Mills at higher loops: some potential problems with integrability*, *JHEP* **08** (2006) 027 [hep-th/0603067];
K. Okamura and K. Yoshida, *Higher loop Bethe ansatz for open spin-chains in AdS/CFT*, *JHEP* **09** (2006) 081 [hep-th/0604100].

[12] N. Mann and S.E. Vazquez, *Classical open string integrability*, *JHEP* **04** (2007) 065 [hep-th/0612038].

[13] I.V. Cherednik, *Factorizing particles on a half line and root systems*, *Theor. Math. Phys.* **61** (1984) 977 [*Teor. Mat. Fiz.* **61** (1984) 35].

[14] S. Ghoshal and A.B. Zamolodchikov, *Boundary S matrix and boundary state in two-dimensional integrable quantum field theory*, *Int. J. Mod. Phys.* **A 9** (1994) 3841 [*Erratum ibid.* **A 9** (1994) 4353] [hep-th/9306002].

[15] K. Iohara and Y. Koga, *Central extensions of Lie superalgebras*, *Comment. Math. Helv.* **76** (2001) 110.

[16] P.P. Kulish and N.Yu. Reshetikhin, *Quantum linear problem for the sine-Gordon equation and higher representation*, *J. Sov. Math.* **23** (1983) 2435 [*Zap. Nauchn. Semin.* **101** (1981) 101].

[17] A.B. Zamolodchikov, *Fractional-spin integrals of motion in perturbed conformal field theory*, in H. Guo, Z. Qiu and H. Tye eds., *Fields, strings and quantum gravity*, Gordon and Breach, (1989).

[18] L. Mezincescu and R.I. Nepomechie, *Fractional-spin integrals of motion for the boundary sine-Gordon model at the free fermion point*, *Int. J. Mod. Phys.* **A 13** (1998) 2747 [hep-th/9709078];
G.W. Delius and N.J. MacKay, *Quantum group symmetry in sine-Gordon and affine Toda field theories on the half-time*, *Commun. Math. Phys.* **233** (2003) 173 [hep-th/0112023].

[19] B.S. Shastry, *Decorated star-triangle relations and exact integrability of the one-dimensional Hubbard model*, *J. Statist. Phys.* **50** (1988) 57.

[20] H.Q. Zhou, *Quantum integrability for the one-dimensional Hubbard open chain*, *Phys. Rev.* **54** (1996) 41;
X.-W. Guan, M.-S. Wang and S.-D. Yang, *Lax pair and boundary K-matrices for the one-dimensional Hubbard model*, *Nucl. Phys.* **485** (1997) 685;
M. Shiroishi and M. Wadati, *Integrable boundary conditions for the one-dimensional Hubbard model*, *J. Phys. Soc. Japan* **66** (1997) 2288 [cond-mat/9708011].

[21] M.J. Martins and C.S. Melo, *The Bethe ansatz approach for factorizable centrally extended S-matrices*, *Nucl. Phys.* **B 785** (2007) 246 [hep-th/0703086].

[22] R.A. Janik, *The $AdS_5 \times S^5$ superstring worldsheet S-matrix and crossing symmetry*, *Phys. Rev.* **D 73** (2006) 086006 [hep-th/0603038].

[23] G. Arutyunov and S. Frolov, *On string S-matrix, bound states and TBA*, *JHEP* **12** (2007) 024 [arXiv:0710.1568].

[24] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [hep-th/0406256].

[25] G. Arutyunov and S. Frolov, *On $AdS_5 \times S^5$ string S-matrix*, *Phys. Lett.* **B 639** (2006) 378 [hep-th/0604043].

[26] N. Beisert, R. Hernandez and E. Lopez, *A crossing-symmetric phase for $AdS_5 \times S^5$ strings*, *JHEP* **11** (2006) 070 [hep-th/0609044].

[27] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and crossing*, *J. Stat. Mech.* (2007) P01021 [hep-th/0610251].

[28] N. Dorey, D.M. Hofman and J.M. Maldacena, *On the singularities of the magnon S-matrix*, *Phys. Rev.* **D 76** (2007) 025011 [hep-th/0703104].

[29] H.-Y. Chen and D.H. Correa, *Comments on the boundary scattering phase*, *JHEP* **02** (2008) 028 [arXiv:0712.1361].

[30] C. Ahn, D. Bak and S.-J. Rey, *Reflecting magnon bound states*, *JHEP* **04** (2008) 050 [arXiv:0712.4144].

[31] E.K. Sklyanin, *Boundary conditions for integrable quantum systems*, *J. Phys.* **A 21** (1988) 2375.

[32] H.-Y. Chen, N. Dorey and K. Okamura, *On the scattering of magnon boundstates*, *JHEP* **11** (2006) 035 [hep-th/0608047].